Univerzitet u Zenici
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Odsjek: Matematika i informatika
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Prvi parcijalni pismeni ispit iz Linearne algebre

Pravila: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. U prostoru $\mathbb{R}^5$ zadan je podprostor $\mathcal{M}$ razapet (generisan) vektorima $(0,0,1,0,0)^\top$ i $(0,1,0,1,0)^\top$ i podprostor

$$\mathcal{L} = \{(x_1, x_2, x_3, x_4, x_5)^\top \in \mathbb{R}^5 \,|\, x_1 - x_2 + x_3 = 0,\ 2x_1 - 2x_2 + x_3 + x_4 = 0\}$$

(a) Odrediti bazu i dimenziju vektorskih prostora $\mathcal{M}$ i $\mathcal{L}$.
(b) Odrediti bazu i dimenziju vektorskih prostora $\mathcal{M} \cap \mathcal{L}$ i $\mathcal{M} + \mathcal{L}$.

2. Zadan je linearni operator $T : \mathrm{Mat}_{2\times2}(\mathbb{R}) \longrightarrow \mathrm{Mat}_{2\times2}(\mathbb{R})$ sa

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & -a+b+2c \\ a-c-d & -a+2c+d \end{bmatrix}.$$

(a) Odrediti po jednu bazu za $\ker(T)$ i $\mathrm{im}(T)$.
(b) Odredite matricu koordinata od T u odnosu na standardnu bazu prostora $\mathrm{Mat}_{2\times2}(\mathbb{R})$.

3. Neka je T linearan operator na prostoru $\mathbb{R}^2$ koji vektor najprije reflektuje (zrcali) s obzirom na pravac $y = -x$, zatim ga rotira za ugao $\frac{\pi}{4}$ oko koordinatnog početka (oko izvorišta) u negativnom smjeru, te zatim reflektuje (zrcali) s obzirom na pravac $y = x$. Naći matricu (matricu koordinata) operatora T u bazi $\mathcal{B} = \left\{2\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}, -\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

4. Neka je $\mathcal{M} = \mathbb{R}^n$ i neka je d funkcija sa $\mathcal{M} \times \mathcal{M}$ u $\mathbb{R}$ definisana sa $d(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, ..., |x_n - y_n|\}$. Provjeriti da li je $(\mathcal{M}, d)$ metrički prostor. Za slučaj kada je $n = 3$ grafički prikazati kugle $B(\mathbf{1}; 1)$ i $B(\mathbf{0}; 2)$.

(#) U prostoru $\mathbb{R}^5$ zadan je podprostor $\mathcal{M}$ razapet (generisan) vektorima $(0,0,1,0,0)^T$ i $(0,1,0,1,0)^T$ i podprostor

$$\mathcal{L} = \{(x_1,x_2,x_3,x_4,x_5)^T \in \mathbb{R}^5 \mid x_1-x_2+x_3=0,\ 2x_1-2x_2+x_3+x_4=0\}.$$

(a) Odrediti bazu i dimenziju vektorskih prostora $\mathcal{M}$ i $\mathcal{L}$.

(b) Odrediti dimenziju vektorskog prostora $\mathcal{M}\cap\mathcal{L}$,

(c) Odrediti neku bazu za (direktni) kompliment prostora $\mathcal{L}$ (koji nije ortogonalni komplement).

Rj.

Red vektor $(x_1,x_2,x_3,x_4,x_5)^T$ ćemo u rješenju pisati kao kolona vektor $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$. Prema postavci zadatka imamo

$$\mathcal{M} = \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}\right\}$$

Dalje

$$\mathcal{L} = \left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}\in\mathbb{R}^5 \;\middle|\; \underbrace{\begin{pmatrix}1&-1&1&0&0\\2&-2&1&1&0\end{pmatrix}}_{=A}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\right\} = \ker A$$

Generator skup za $\ker A$ su vektori iz opšteg rješenja sistema $Ax=0$

$$\begin{pmatrix}1&-1&1&0&0\\2&-2&1&1&0\end{pmatrix} \overset{II_v+I_v\cdot(-2)}{\sim} \begin{pmatrix}1&-1&1&0&0\\0&0&-1&1&0\end{pmatrix} \Rightarrow \operatorname{rang}A = 2 = \operatorname{rang}\bar{A}$$

Sistem Ax ima ∞ mnogo rješenja i 3 promjenjive uzimamo proizvoljno npr. $x_2=s$, $x_4=t$, $x_5=u$

$x_1 = x_2 - x_3 \Rightarrow x_1 = s-t$

$-x_3 = -x_4 \Rightarrow x_3 = t$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}s-t\\s\\t\\t\\u\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}s + \begin{pmatrix}-1\\0\\1\\1\\0\end{pmatrix}t + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}u$$

Time je $\mathcal{L} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Prema tome $\dim \mathcal{M} = 2$, $\dim \mathcal{L} = 3$ a baze za $\mathcal{M}$ i $\mathcal{L}$ su redom
$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad i \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

b) Prisjetimo se:

<u>Dimenzija sume</u>

<u>Ako su $\mathcal{X}$ i $\mathcal{Y}$ podprostori vektorskog prostora V, tada</u>

$$\underline{\dim(\mathcal{X} + \mathcal{Y}) = \dim(\mathcal{X}) + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y})}$$

<u>gdje je $\mathcal{X} + \mathcal{Y} = \{ x + y \mid x \in \mathcal{X} \text{ i } y \in \mathcal{Y} \}$.</u>

Ako sa $\mathcal{B}_{\mathcal{M}}$ označimo bazu za $\mathcal{M}$ a sa $\mathcal{B}_{\mathcal{L}}$ označimo bazu za $\mathcal{L}$, vidimo da $\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}_{\mathcal{L}}$ generira $\mathcal{M} + \mathcal{L}$.

Dimenziju za $\mathcal{M} + \mathcal{L}$ nije teško odrediti, posmatramo broj linearno nezavisnih kolona iz $\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}_{\mathcal{L}}$:

$$D = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{I_k \Leftrightarrow III_k}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{II_v + I_v(-1)}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{IV_v + II_v \cdot (-1)}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\operatorname{rang} D = 4 \Rightarrow \dim(\mathcal{X} + \mathcal{Y}) = 4$$

$$\| $$

$$\dim \underset{2}{\mathcal{X}} + \dim \underset{3}{\mathcal{Y}} - \dim(\mathcal{X} \cap \mathcal{Y})$$

$$\dim(\mathcal{X} \cap \mathcal{Y}) = 1$$

U zadatku se ne traži da odredimo bazu za $\mathcal{X} \cap \mathcal{Y}$. Međutim, ako bi željeli da odredimo ovu bazu prvo primjetimo da je

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

i kako je $\dim(\mathcal{X} \cap \mathcal{Y}) = 1$ to je $\mathcal{X} \cap \mathcal{Y} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

c) Prisjetimo se

<u>Komplementarni podprostori</u>

<u>Za podprostore $\mathcal{X}$ i $\mathcal{Y}$ prostora $\mathcal{V}$ kažemo da su komplementarni kadgod je</u>

$$\underline{\mathcal{V} = \mathcal{X} + \mathcal{Y} \quad i \quad \mathcal{X} \cap \mathcal{Y} = \mathbf{0}}.$$

<u>i u ovom slučaju za $\mathcal{V}$ kažemo da je direktna suma od $\mathcal{X}$ i $\mathcal{Y}$, i ovo označavamo sa $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$.</u>

<u>Ako su $\mathcal{B}_{\mathcal{X}}$ i $\mathcal{B}_{\mathcal{Y}}$ baze za $\mathcal{X}$ i $\mathcal{Y}$ tada</u>

$\underline{\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}}$ akko $\underline{\forall v \in \mathcal{V}\ \exists!\, x \in \mathcal{X}, y \in \mathcal{Y} \text{ t.d. } v = x + y}$ akko $\underline{\mathcal{B}_{\mathcal{X}} \cap \mathcal{B}_{\mathcal{Y}} = \emptyset \text{ i } \mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}} \text{ je baza za } \mathcal{V}}$

Ako sa B označimo matricu

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

tada je

$$\operatorname{im}(B^T) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathcal{L}$$

pa produžimo bazu od $\mathcal{L}$ do baze prostora $\mathbb{R}^5$.

Znamo da je, za proizvoljne matrice A i B

$$\underline{\operatorname{im}(A^T) = \operatorname{im}(B^T) \text{ akko } A \overset{red}{\sim} B}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[IV_v + I_v \cdot (-1)]{II_v + I_v} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[V_v + II_v \cdot (-1)]{IV_v + II_v} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Baza za direktni komplement prostora $\mathcal{L}$ je

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(#) Zadan je linearni operator $T: Mat_{2\times 2}(\mathbb{R}) \to Mat_{2\times 2}(\mathbb{R})$ sa

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & -a+b+2c \\ a-c-d & -a+2c+d \end{bmatrix}.$$

(a) Odredite po jednu bazu za $\ker(T)$ i $im(T)$.

(b) Odredite matricu koordinata od T u odnosu na standardnu bazu prostora $Mat_{2\times 2}(\mathbb{R})$.

Rj.:

(a) Za jednostavniji pristup umjesto prostora $Mat_{2\times 2}(\mathbb{R})$ posmatrajmo prostor $\mathbb{R}^4$ i operator $T': \mathbb{R}^4 \to \mathbb{R}^4$ definisan sa

$$T'\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{pmatrix} a-b \\ -a+b+2c \\ a-c-d \\ -a+2c+d \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\ker(T') = \{ x \mid T'(x) = 0 \} = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \,\middle|\, A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mathbf{0} \right\} = \ker(A)$$

Znamo:

<u>Jezgro od A je opšte rješenje sistema $Ax = \mathbf{0}$.</u>

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 2 & 1 & 0 \end{array}\right] \underset{\substack{III_v - I_v \\ IV_v + I_v}}{\overset{II_v + I_v}{\sim}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 2 & 1 & 0 \end{array}\right] \overset{II_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right] \overset{III_v + II_v}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right] \overset{IV_v + III_v \cdot (-2)}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} a-d=0 \\ b-d=0 \\ c=0 \\ \hline d\in\mathbb{R} \end{array} \qquad \begin{pmatrix} x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix} s\\s\\0\\s\end{pmatrix} = \begin{pmatrix}1\\1\\0\\1\end{pmatrix} s$$

baza za $\ker(T)$

Prema tome $\ker(T) = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

$$\text{im}(T) = \{ T(x) \mid x \in \mathbb{R}^4 \} = \left\{ A\begin{pmatrix}a\\b\\c\\d\end{pmatrix} \;\middle|\; \begin{pmatrix}a\\b\\c\\d\end{pmatrix} \in \mathbb{R}^4 \right\} = \text{im}(A)$$

Znamo:

<u>Generator skup za im(A) su osnovne kolone u A.</u>

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

baze za $\text{im}(T)$.

$$\Rightarrow \quad \text{im}(T) = \text{span}\left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \right\}$$

(b) Standardna baza za $Mat_{2\times 2}$ je $\left\{ \begin{bmatrix} 1&0\\0&0 \end{bmatrix}, \begin{bmatrix} 0&1\\0&0 \end{bmatrix}, \begin{bmatrix} 0&0\\1&0 \end{bmatrix}, \begin{bmatrix} 0&0\\0&1 \end{bmatrix} \right\}$

$$[T]_\varphi = \left(\left[T\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]_\varphi \; \left[T\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]_\varphi \; \left[T\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]_\varphi \; \left[T\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]_\varphi \right)$$

$$T\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&-1\\1&-1\end{pmatrix}, \quad T\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}-1&1\\0&0\end{pmatrix}, \quad T\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&2\\-1&2\end{pmatrix}, \quad T\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\-1&1\end{pmatrix}$$

$$[T]_\varphi = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

(#) Neka je T linearan operator na prostoru $\mathbb{R}^2$ koji vektor najprije reflektuje (zrcali) s obzirom na pravac $y=-x$, zatim ga rotira za ugao $\frac{\pi}{4}$ oko koordinatnog početka (oko ishodišta) u negativnom smjeru, te zatim reflektuje (zrcali) s obzirom na pravac $y=x$. Naći matricu (matricu koordinata) operatora T u bazi $\mathcal{B}=\left\{2\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}, -\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

Rj: Prisjetimo se

Matrica koordinata

Neka su $\mathcal{B}=\{u_1,u_2,\dots,u_n\}$ i $\mathcal{B}'=\{v_1,v_2,\dots,v_m\}$, redom, baze za $\mathcal{U}$ i $\mathcal{V}$. Matrica koordinata od $T\in\mathcal{L}(\mathcal{U},\mathcal{V})$ u odnosu na par $(\mathcal{B},\mathcal{B}')$ je definisana kao $m\times n$ matrica

$$[T]_{\mathcal{B}\mathcal{B}'}=\begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

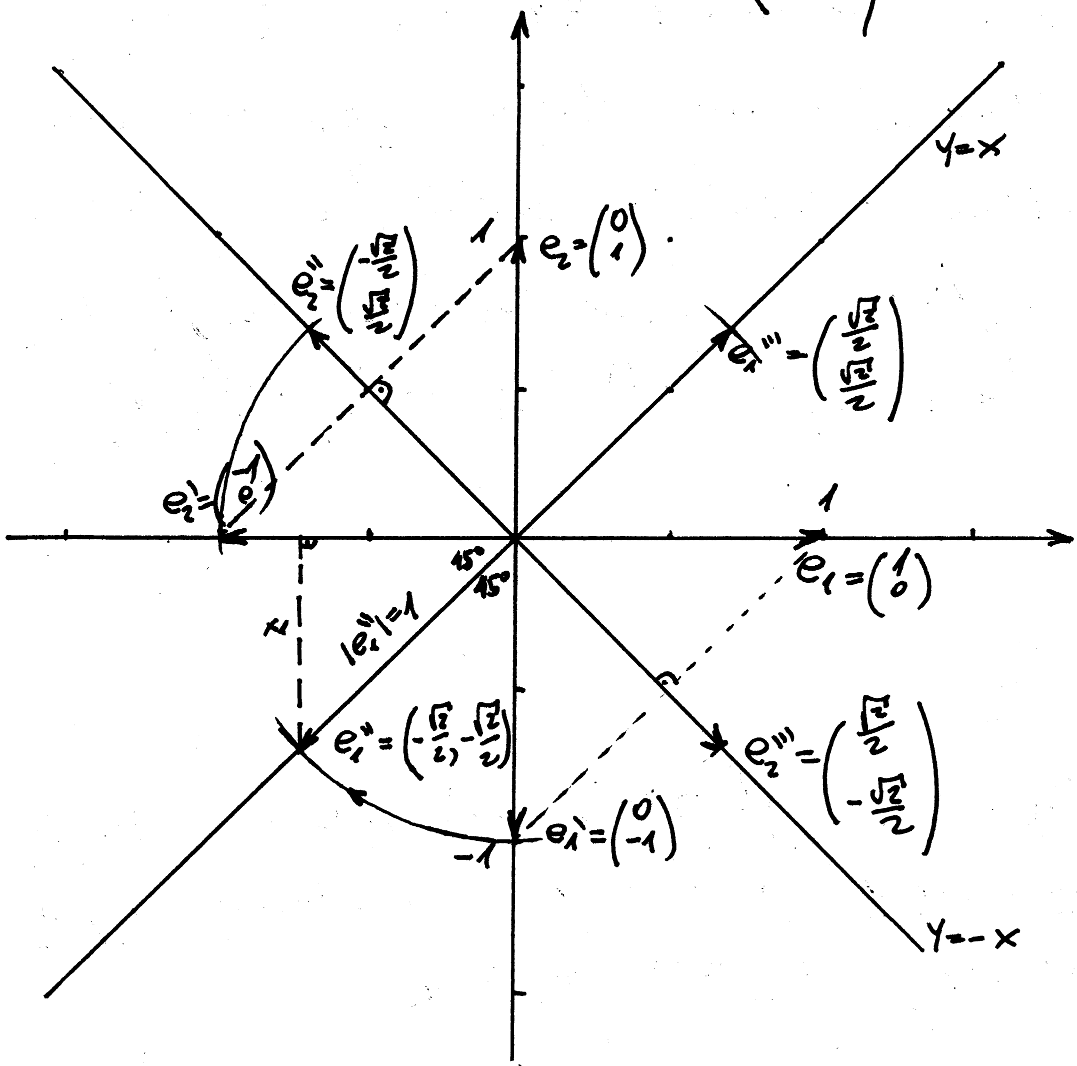
Kada je T linearni operator na $\mathcal{U}$, tada je u igri samo jedna baza, i koristimo $[T]_{\mathcal{B}}$ umjesto $[T]_{\mathcal{B}\mathcal{B}}$.

U našem slučaju $\mathcal{B}=\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}$.

Puno je lakše prvo odrediti matricu linearnog operatora u odnosu na standardnu bazu $\mathcal{S}=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

$$\varphi = \{\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{e_2}\}$$

$$[T]_\varphi = \begin{pmatrix} | & | \\ [T(e_1)]_\varphi & [T(e_2)]_\varphi \\ | & | \end{pmatrix}$$



$\sin 45° = \frac{x_1}{1}$

$x_1 = \frac{\sqrt2}{2}$

Sa slike nije teško izračunati da je

$$[T(e_1)]_\varphi = \underbrace{\begin{pmatrix}\frac{\sqrt2}{2}\\ \frac{\sqrt2}{2}\end{pmatrix}}_{=T(e_1)}, \qquad [T(e_2)]_\varphi = \underbrace{\begin{pmatrix}\frac{\sqrt2}{2}\\ -\frac{\sqrt2}{2}\end{pmatrix}}_{=T(e_2)}$$

Primjetimo se

Djelovanje kao matrično množenje

Neka je $T \in \mathcal{L}(U,V)$ i neka su $\mathcal{B}$ i $\mathcal{B}'$ baze za U i V redom. Za $\forall u \in U$

$$[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'} \cdot [u]_{\mathcal{B}}$$

Nama treba matrica koordinata operatora T u bazi $\mathcal{B}$

$$[T]_{\mathcal{B}} = \begin{pmatrix} | & | \\ \left[T\begin{pmatrix}2\\-1\end{pmatrix}\right]_{\mathcal{B}} & \left[T\begin{pmatrix}-1\\2\end{pmatrix}\right]_{\mathcal{B}} \\ | & | \end{pmatrix}$$

Ako vektore baze $\mathcal{B} = \left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}$ označimo sa u_1 i u_2

tj. sa $u_1 = \begin{pmatrix}2\\-1\end{pmatrix}$, $u_2 = \begin{pmatrix}-1\\2\end{pmatrix}$, da bi odredili $[T(u_1)]_{\mathcal{B}}$ i $[T(u_2)]_{\mathcal{B}}$

koristimo formule

$$[T(u_1)]_{\mathcal{B}} = [T]_{\varphi\mathcal{B}} \cdot [u_1]_{\varphi}$$

$$[T(u_2)]_{\mathcal{B}} = [T]_{\varphi\mathcal{B}} \cdot [u_2]_{\varphi}$$

Znamo $[u_1]_{\varphi} = \begin{pmatrix}2\\-1\end{pmatrix}$ i $[u_2]_{\varphi} = \begin{pmatrix}-1\\2\end{pmatrix}$. Odredimo još $[T]_{\varphi\mathcal{B}}$

$$[T]_{\varphi\mathcal{B}} = \begin{pmatrix} | & | \\ [T(e_1)]_{\mathcal{B}} & [T(e_2)]_{\mathcal{B}} \\ | & | \end{pmatrix}$$

Tražimo α i β t.d. $\alpha\begin{pmatrix}2\\-1\end{pmatrix} + \beta\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} \Rightarrow$

$\Rightarrow$ ZA VJEŽBU ... $\Rightarrow$ $\alpha = \frac{\sqrt{2}}{2}$, $\beta = \frac{\sqrt{2}}{2}$ $\Rightarrow$ $[T(e_1)]_{\mathcal{B}} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$

Pa sad tražimo npr γ i δ t.d. $\gamma\begin{pmatrix}2\\-1\end{pmatrix} + \delta\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix} \Rightarrow$

$\Rightarrow$ ZA VJEŽBU ... $\Rightarrow$ $\gamma = \frac{\sqrt{2}}{6}$, $\beta = \frac{-\sqrt{2}}{6}$ $\Rightarrow$

$$\Rightarrow \quad [T(e_2)]_{\mathcal{B}} = \begin{pmatrix} \frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{6} \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad [T]_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \end{pmatrix}$$

Na kraju imamo

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{5\sqrt{2}}{6} \\ \frac{7\sqrt{2}}{6} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{6} \\ -\frac{5\sqrt{2}}{6} \end{pmatrix}$$

Prema tome

$$[T]_{\mathcal{B}} = \begin{pmatrix} \frac{5\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} \\ \frac{7\sqrt{2}}{6} & -\frac{5\sqrt{2}}{6} \end{pmatrix}$$